

Use of MAX-CUT for Ramsey Arrowing of Triangles

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Abstract

In 1967, Erdős and Hajnal asked the question: Does there exist a K_4 -free graph that is not the union of two triangle-free graphs? Finding such a graph involves solving a special case of the classical Ramsey arrowing operation. Folkman proved the existence of these graphs in 1970, and they are now called Folkman graphs. Erdős offered \$100 for deciding if one exists with less than 10^{10} vertices. This problem remained open until 1988 when Spencer, in a seminal paper using probabilistic techniques, proved the existence of a Folkman graph of order 3×10^9 (after an erratum), without explicitly constructing it. In 2008, Dudek and Rödl developed a strategy to construct new Folkman graphs by approximating the maximum cut of a related graph, and used it to improve the upper bound to 941. We improve this bound first to 860 using their approximation technique and then further to 786 with the MAX-CUT semidefinite programming relaxation as used in the Goemans-Williamson algorithm.

1 Introduction

Given a simple graph G , we write $G \rightarrow (a_1, \dots, a_k)^e$ and say that G *arrows* $(a_1, \dots, a_k)^e$ if for every edge k -coloring of G , a monochromatic K_{a_i} is forced for some color $i \in \{1, \dots, k\}$. Likewise, for graphs F and H , $G \rightarrow (F, H)^e$ if for every edge 2-coloring of G , a monochromatic F is forced in the first color or a monochromatic H is forced in the second. Define $\mathcal{F}_e(a_1, \dots, a_k; p)$ to be the set of all graphs that arrow (a_1, \dots, a_k) and do not contain K_p ; they are often called Folkman graphs. The edge Folkman number $F_e(a_1, \dots, a_k; p)$ is the smallest order of a graph that is a member of $\mathcal{F}_e(a_1, \dots, a_k; p)$. In 1970, Folkman [6] showed that for $k > \max\{s, t\}$, $F_e(s, t; k)$ exists. The related problem of vertex Folkman numbers, where vertices are colored instead of edges, is more studied [15, 17] than edge Folkman numbers, but we will not be discussing them.

In 1967, Erdős and Hajnal [5] asked the question: Does there exist a K_4 -free graph that is not the union of two triangle-free graphs? This question is equivalent to asking for the existence of a K_4 -free graph such that in any edge 2-coloring, a monochromatic triangle is forced. After Folkman proved the existence of such a graph, the question then became to find how small this graph could be, or using the above notation, what the value of $F_e(3, 3; 4)$ is. Prior to this paper, the best known bounds for this case were $19 \leq F_e(3, 3; 4) \leq 941$ [20, 4].

Folkman numbers are related to Ramsey numbers $R(s, t)$, defined as the least positive n such that any 2-coloring of the edges of K_n yields a monochromatic K_s in the first color or a monochromatic K_t in the second. Using the arrowing operator, it is clear that $R(s, t)$ is the smallest n such that $K_n \rightarrow (s, t)^e$. The known values and bounds for various types of Ramsey numbers are collected and regularly updated by the second author [19].

We will be using standard graph theory notation: $V(G)$ and $E(G)$ for the vertex and edge sets of graph G , respectively. A *cut* is a partition of the vertices of a graph into two sets, $S \subset V(G)$ and $\overline{S} = V(G) \setminus S$. The *size* of a cut is the number of edges that join the two sets, that is, $|\{\{u, v\} \in E(G) \mid u \in S \text{ and } v \in \overline{S}\}|$. MAX-CUT is a well-known **NP**-hard combinatorial optimization problem which asks for the maximum size of a cut of a graph.

2 History of $F_e(3, 3; 4)$

Year	Lower/Upper Bounds	Who/What	Ref.
1967	any?	Erdős-Hajnal	[5]
1970	exist	Folkman	[6]
1972	10 –	Lin	[12]
1975	– 10^{10} ?	Erdős offers \$100 for proof	
1986	– 8×10^{11}	Frankl-Rödl	[7]
1988	– 3×10^9	Spencer	[22]
1999	16 –	Piwakowski et al. (implicit)	[18]
2007	19 –	Radziszowski-Xu	[20]
2008	– 9697	Lu	[14]
2008	– 941	Dudek-Rödl	[4]
2012	– 786	this work	
2012	– 100?	Graham offers \$100 for proof	

Table 1: Timeline of progress on $F_e(3, 3; 4)$.

Table 1 summarizes the events surrounding $F_e(3, 3; 4)$, starting with Erdős and Hajnal’s [5] original question of existence. After Folkman [6] proved the existence, Erdős, in 1975, offered \$100 for deciding if $F_e(3, 3; 4) < 10^{10}$. This question remained open for over 10 years. Frankl and Rödl [7] nearly met Erdős’ request in 1986 when they showed that $F_e(3, 3; 4) < 7.02 \times 10^{11}$. In 1988, Spencer [22], in a seminal paper using probabilistic techniques, proved the existence of a Folkman graph of order 3×10^9 (after an erratum by Hovey), without explicitly constructing it. In 2007, Lu showed that $F_e(3, 3; 4) \leq 9697$ by constructing a family of K_4 -free circulant graphs (which we discuss in Section 3.3) and showing that some such graphs arrow $(3, 3)^e$ using spectral analysis. Later, Dudek and Rödl reduced the upper bound to the best known to date, 941. Their method, which we have pursued further with some success, is discussed in the next section.

The lower bound for $F_e(3, 3; 4)$ was much less studied than the

upper bound. Lin [12] obtained a lower bound on 10 in 1972 without the help of a computer. All 659 graphs on 15 vertices witnessing $F_e(3, 3; 5) = 15$ [18] contain K_4 , thus giving the bound $16 \leq F_e(3, 3; 4)$. In 2007, two of the authors of this paper gave a computer-free proof of $18 \leq F_e(3, 3; 4)$ and improved the lower bound further to 19 with the help of computations [20].

The long history of $F_e(3, 3; 4)$ is not only interesting in itself but also gives insight into how difficult the problem is. Finding good bounds on the smallest order of any Folkman graph (with fixed parameters) seems to be difficult, and some related Ramsey graph coloring problems are **NP**-hard or lie even higher in the polynomial hierarchy. For example, Burr [2] showed that arrowing $(3, 3)^e$ is **coNP**-complete, and Schaefer [21] showed that for general graphs F , G , and H , $F \rightarrow (G, H)$ is Π_2^P -complete.

3 Arrowing via MAX-CUT

Building off Spencer's and other methods, Dudek and Rödl [4] in 2008 showed how to construct a graph H_G from a graph G , such that the maximum size of a cut of H_G determines whether or not $G \rightarrow (3, 3)^e$. They construct the graph H_G as follows. The vertices of H_G are the edges of G , so $|V(H_G)| = |E(G)|$. For $e_1, e_2 \in V(H_G)$, if edges $\{e_1, e_2, e_3\}$ form a triangle in G , then $\{e_1, e_2\}$ is an edge in H_G .

Let $t_\Delta(G)$ denote the number of triangles in graph G . Clearly, $|E(H_G)| = 3t_\Delta(G)$. Let $MC(H)$ denote the MAX-CUT value of graph H .

Theorem 1 (Dudek and Rödl [4]). *$G \rightarrow (3, 3)^e$ if and only if $MC(H_G) < 2t_\Delta(G)$.*

There is a clear intuition behind Theorem 1 that we will now describe. Any edge 2-coloring of G corresponds to a bipartition of the vertices in H_G . If a triangle colored in G is not monochromatic, then its three edges which are vertices in H_G will be separated in the bipartition. If we treat this bipartition as a cut, then the size of the cut will count each triangle twice for the two edges that cross it.

Since there is only one triangle in a graph that contains two given edges, this effectively counts the number of non-monochromatic triangles. Therefore, if it is possible to find a cut that has size equal to $2t_\Delta(G)$, then such a cut defines an edge coloring of G that has no monochromatic triangles. However, if $MC(H_G) < 2t_\Delta(G)$, then in each coloring, all three edges of some triangle are in one part and thus, $G \rightarrow (3, 3)^e$.

A benefit of converting the problem of arrowing $(3, 3)^e$ to MAX-CUT is that the latter is well-known and has been studied extensively in computer science and mathematics (see for example [3]). The decision problem MAX-CUT(H, k) asks whether or not $MC(H) \geq k$. It is known that MAX-CUT is **NP**-hard and this decision problem was one of Karp's 21 **NP**-complete problems [11]. In our case, $G \rightarrow (3, 3)^e$ if and only if MAX-CUT($H_G, 2t_\Delta(G)$) doesn't hold. Since MAX-CUT is **NP**-hard, an attempt is often made to approximate it, such as in the approaches presented in the next two sections.

3.1 Minimum Eigenvalue Method

A method exploiting the minimum eigenvalue was used by Dudek and Rödl [4] to show that some large graphs are members of $\mathcal{F}_e(3, 3; 4)$. The following upper bound (1) on $MC(H_G)$ can be found in [4], where λ_{\min} denotes the minimum eigenvalue of the adjacency matrix of H_G .

$$MC(H_G) \leq \frac{|E(H_G)|}{2} - \frac{\lambda_{\min}|V(H_G)|}{4}. \quad (1)$$

For positive integers r and n , if -1 is an r -th residue modulo n , then let $G(n, r)$ be a circulant graph on n vertices with the vertex set \mathbb{Z}_n and the edge set $E(G(n, r)) = \{\{u, v\} \mid u \neq v \text{ and } u - v \equiv \alpha^r \pmod{n}, \text{ for some } \alpha \in \mathbb{Z}_n\}$.

The graph $G_{941} = G(941, 5)$ has 707632 triangles. Using the MATLAB [16] `eigs` function, Dudek and Rödl [4] computed

$$MC(H_{G_{941}}) \leq 1397484 < 1415264 = 2t_\Delta(G_{941}).$$

Thus, by Theorem 1, $G_{941} \rightarrow (3, 3)^e$.

In an attempt to improve over $F_e(3, 3; 4) \leq 941$, we tried removing vertices of G_{941} to see if the minimum eigenvalue bound would

still show arrowing. We applied multiple strategies for removing sets of vertices and most were successful. This led to the following theorem:

Theorem 2. $F_e(3, 3; 4) \leq 860$.

Proof. For a graph G with vertices \mathbb{Z}_n , define $C = C(d, k) = \{v \in V(G) \mid v = id \bmod n, \text{ for } 0 \leq i < k\}$. Let $G = G_{941}$, $d = 2$, $k = 81$, and G_C be the graph induced on $V(G) \setminus C(d, k)$. Then G_C has 860 vertices, 73981 edges and 542514 triangles. Using the upper bound (1) and the MATLAB `eigs` function, we obtain

$$MC(H_{G_C}) \leq 1084967 < 1085028 = 2t_\Delta(G_C). \quad (2)$$

Therefore, $G_C \rightarrow (3, 3)^e$. \square

None of the methods used allowed for 82 or more vertices to be removed without the upper bound on MC becoming larger than $2t_\Delta$.

3.2 Goemans-Williamson Method

The Goemans-Williamson MAX-CUT approximation algorithm [8] is a well-known, polynomial-time algorithm that relaxes the problem to a semidefinite program (SDP). It involves the first use of SDP in combinatorial approximation and has since inspired a variety of other successful algorithms (see for example [13]). This randomized algorithm returns a cut with expected size at least 0.87856 of the optimal value. However, in our case, all that is needed is the solution to the SDP, as it gives an upper bound on $MC(H)$. A brief description of the Goemans-Williamson relaxation follows.

The first step in relaxing MAX-CUT is to represent the problem as a quadratic integer program. Given a graph H with $V(H) = \{1, \dots, n\}$ and nonnegative weights $w_{i,j}$ for each pair of vertices $\{i, j\}$, we can write $MC(H)$ as the following objective function:

$$\begin{aligned} &\text{Maximize} \quad \frac{1}{2} \sum_{i < j} w_{i,j} (1 - y_i y_j) \\ &\text{subject to:} \quad y_i \in \{-1, 1\} \quad \text{for all } i \in V(H). \end{aligned} \quad (3)$$

Define one part of the cut as $S = \{i \mid y_i = 1\}$. Since in our case all graphs are weightless, we will use

$$w_{i,j} = \begin{cases} 1 & \text{if } \{i, j\} \in E(H), \\ 0 & \text{otherwise.} \end{cases}$$

Next, the integer program (3) is relaxed by extending the problem to higher dimensions. Each $y_i \in \{-1, 1\}$ is now replaced with a vector on the unit sphere $\mathbf{v}_i \in \mathbb{R}^n$, as follows:

$$\begin{aligned} & \text{Maximize} \quad \frac{1}{2} \sum_{i < j} w_{i,j} (1 - \mathbf{v}_i \cdot \mathbf{v}_j) \\ & \text{subject to:} \quad \|\mathbf{v}_i\| = 1 \quad \text{for all } i \in V(H). \end{aligned} \tag{4}$$

If we define a matrix Y with the entries $y_{i,j} = \mathbf{v}_i \cdot \mathbf{v}_j$, that is, the Gram matrix of $\mathbf{v}_1, \dots, \mathbf{v}_n$, then $y_{i,i} = 1$ and Y is positive semi-definite. Therefore, (4) is a semidefinite program.

3.3 Some Cases of Arrowing

Using the Goemans-Williamson approach, we tested a wide variety of graphs for arrowing by finding upper bounds on MAX-CUT. The type of graph that led to the best results was described by Lu [14].

For positive integers n and s , $s < n$, s relatively prime to n , define set $S = \{s^i \bmod n \mid i = 0, 1, \dots, m-1\}$, where m is the smallest positive integer such that $s^m \equiv 1 \bmod n$. If $-1 \bmod n \in S$, then let $L(n, s)$ be a circulant graph on n vertices with $V(L(n, s)) = \mathbb{Z}_n$. For vertices u and v , $\{u, v\}$ is an edge of $L(n, s)$ if and only if $u - v \in S$. Note that the condition that $-1 \bmod n \in S$ implies that if $u - v \in S$ then $v - u \in S$.

In Table 1 of [14], a set of potential members of $\mathcal{F}_e(3, 3; 4)$ of the form $L(n, s)$ were listed, and the graph $L(9697, 4)$ was shown to arrow $(3, 3)^e$. Lu gave credit to Exoo for showing that $L(17, 2)$, $L(61, 8)$, $L(79, 12)$, $L(421, 7)$, and $L(631, 24)$ do not arrow $(3, 3)^e$.

We tested all graphs from Table 1 of [14] with order less than 941 with the MAX-CUT method, using both the minimum eigenvalue

G	$2t_{\Delta}(G)$	λ_{\min}	SDP
$L(127, 5)$	19558	20181	20181
$L(457, 6)$	347320	358204	358204
$L(761, 3)$	694032	731858	731858
$L(785, 53)$	857220	857220	857220
G_{786}	857762	857843	857753

Table 2: Potential $\mathcal{F}_e(3, 3; 4)$ graphs G and upper bounds on $MC(H_G)$, where “ λ_{\min} ” is the bound (1) and “SDP” is the solution of (4) from SDPLR-MC, SDPLR, and SBmethod. G_{786} is the graph of Theorem 3.

and SDP upper bounds. Multiple SDP solvers that were designed to handle large-scale SDP and MAX-CUT problems were used. Specifically, we made use of two versions of SDPLR by Samuel Burer [1], both using low-rank factorization. SDPLR-MC is a version of the software specifically for the MAX-CUT relaxation. The regular software SDPLR is meant for any SDP. SBmethod by Christoph Helmberg [10] implements a spectral bundle method and was also applied successfully in our experiments. Table 2 lists the results. In all cases, all three solvers gave the same result.

Note that although none of the computed upper bounds of the $L(n, s)$ graphs imply arrowing $(3, 3)^e$, all SDP bounds match those of the minimum eigenvalue bound. This is distinct from other families of graphs, including those in [4], as the SDP bound is usually tighter. Thus, these graphs were given further consideration.

$L(127, 5)$ was given particular attention, as it is the same graph as G_{127} , where $V(G_{127}) = \mathbb{Z}_{127}$ and $E(G_{127}) = \{\{x, y\} \mid x - y \equiv \alpha^3 \pmod{127}\}$ (that is, the graph $G(127, 3)$ as defined in the previous section). It has been conjectured by Exoo that $G_{127} \rightarrow (3, 3)^e$. He also suggested that subgraphs induced on less than 100 vertices of G_{127} may as well. For more information on G_{127} see [20].

Numerous attempts were made at modifying these graphs in hopes that one of the MAX-CUT methods would be able to prove arrowing. Indeed, we were able to do so with $L(785, 53)$. Notice that all of the upper bounds for $MC(H_{L(785, 53)})$ are 857220, the same as $2t_{\Delta}(L(785, 53))$. Our goal was then to slightly modify $L(785, 53)$

so that this value becomes smaller. Let G_{786} denote the graph $L(785, 53)$ with one additional vertex connected to the following 60 vertices:

$$\{ 0, 1, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18, 19, 21, 22, 24, 25, 27, \\ 28, 30, 31, 33, 34, 36, 37, 39, 40, 42, 43, 45, 46, 48, 49, 51, 52, \\ 54, 55, 57, 58, 60, 61, 63, 66, 69, 201, 204, 207, 210, 213, 216, \\ 219, 222, 225, 416, 419, 422, 630, 642, 645 \}.$$

G_{786} is still K_4 -free, has 61290 edges and has 428881 triangles. The upper bound computed from all SDP solvers for $MC(H_{G_{786}})$ is 857753. Software implementing **SpeedP** by Grippo et al. [9], an algorithm designed to solve large MAX-CUT SDP relaxations, was used by Rinaldi (one of the authors of [9]) to analyze this graph. He was able to obtain the bounds $857742 \leq MC(H_{G_{786}}) \leq 857750$. Since $2t_{\Delta}(G_{786}) = 857762$, we have both from our tests and the **SpeedP** test that $G_{786} \rightarrow (3, 3)^e$, and the following main result.

Theorem 3. $F_e(3, 3; 4) \leq 786$.

We note that the lower bound $857742 \leq MC(H_{G_{786}})$ followed from finding an actual cut of $H_{G_{786}}$ of size 857742. This method may be useful, as finding a cut of size $2t_{\Delta}(G)$ shows that $G \not\rightarrow (3, 3)^e$.

4 Tasks to Complete

Improving the upper bound on $F_e(3, 3; 4) \leq 786$ is the main challenge. The question of whether $G_{127} \rightarrow (3, 3)^e$ is still open, and any method that could solve it would be of much interest.

During the 2012 SIAM Conference on Discrete Mathematics in Halifax, Nova Scotia, Ronald Graham announced a \$100 award for determining if $F_e(3, 3; 4) < 100$.

Another open question is the lower bound on $F_e(3, 3; 4)$, as it is quite puzzling that only 19 is the best known. Even an improvement to $20 \leq F_e(3, 3; 4)$ would be good progress.

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